As an illustration we examine the problem

$$\frac{\partial U_1}{\partial x} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left[\rho U_1^2 \frac{\partial U_1}{\partial \rho}\right] + \frac{0.5(\rho - R)^2}{[1.5\rho^2 + 1.5x(\rho - R)^2]^{1/3}} - 2(1 + x) - \frac{R}{\rho}x, \quad 0 < \rho < R;$$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left[\rho U_2 \frac{\partial U_2}{\partial \rho}\right] = 0, \quad R < \rho < 2R;$$

$$\frac{\partial U_1(0,\ x)}{\partial \rho} = 0, \quad U_1(\rho,\ 0) = \sqrt[3]{2R^2 \ln 2 + (1.5)^{2/3} R^{4/3}};$$

$$U_1(R - 0,\ x) = U_2(R + 0,\ x), \quad U_2(2R) = \sqrt{2R^2 \ln 2 + (1.5)^{2/3} R^{4/3}};$$

$$\rho U_1^2 \frac{\partial U_1}{\partial \rho}\Big|_{\rho = R - 0} - \rho U_2 \frac{\partial U_2}{\partial \rho}\Big|_{\rho = R + 0} = \Psi(x).$$

It is required to find $U(\rho,\ x)$ and $\Psi(x)$. This problem has the known exact solution

$$U_1 = \sqrt[3]{[1.5\rho^2 + 1.5x(\rho - R)^2]^2},$$

$$U_2 = \sqrt{2R^2 \ln(\rho/R) + (1.5)^{2/3} R^{4/3}}, \quad \Psi(x) = 0.$$

To find an approximate solution we use the difference scheme (27) with $h = 0.1$, $\tau = 0.04$, and $R = 1$. We have carried out the numerical computation on a BÉSM-4 digital computer. We give the values of $Y_{i,k}$ for $K = 400$: $Y_{0,k} = 2.8843$, $Y_{2,k} = 2.4897$, $Y_{4,k} = 2.0715$, $Y_{6,k} = 1.6368$, $Y_{8,k} = 1.2431$, $Y_{10,k} = 1.1442$, $Y_{12,k} = 1.2920$, $Y_{14,k} = 1.4083$, $Y_{16,k} = 1.5004$, $Y_{18,k} = 1.5776$, $Y_{20,k} = 1.6425$. We have also made a comparison of the $Y_{i,k}$ for $K = 400$ with the exact solution for $x = 16$. The error turns out to be not greater than 0.007.

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EFFECT OF A BIPERIODIC SYSTEM OF PLANE INCLUSIONS ON A PLANE STEADY TEMPERATURE FIELD

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Determining the complex potential of a plane temperature field perturbed by a biperiodic system of thin inclusions reduces to the solution of a singular integrodifferential equation.

1. Suppose that a plane steady temperature field is perturbed by some finite system of cuts (lines) Γ_n, $n = \overline{1, N}$. Each line may be taken to be, e.g., a foreign inclusion (or crack) of sufficiently large extension (relative to its width), the thermal conductivity k_n of which differs from the thermal conductivity k of the basic medium, taken to be the complex-variable plane $z = x + iy$. The set of all the lines Γ_n is denoted by $\Gamma = \Gamma_1 + \ldots + \Gamma_N$.

Consider the problem of finding the temperature field perturbed by inclusions, assuming that the temperature in a homogeneous body (in the absence of inclusions) is determined by a given harmonic function $T_0(x, y) = \mathrm{Re}\, F(z)$.

The complex potential of the perturbed temperature field $W(t) = T + i\psi$, where ψ is the current function associated with the temperature T, will be found as the sum of a given function F(z) and a Cauchy-type integral of unknown density taken along the curve Γ

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$$W(z) = F(z) + \Phi(z), \quad \Phi(z) = \frac{1}{2\pi i}\int_{\Gamma}\frac{\mu(t)\,dt}{t-z}. \tag{1.1}$$

Assuming, so as to be specific, that $k_n = k_0 \ll k$, the unknown function $\mu(t)$ will be found using the boundary condition [1]

$$\delta\frac{\partial\psi}{\partial s} = T^- - T^+, \quad \delta = 2h_0k/k_0. \tag{1.2}$$

Here T^+ and T^- are the values of the temperature at the left-hand and right-hand edges of the inclusion; $2h_0(s)$ is the width of the inclusion in the cross section s.

It is assumed that the derivative of $\mu(t)$ is continuous in the Holder sense [2]. Applying the Sokhotskii formula [2] to the function W(z) in Eq. (1.1) as $z \to t \in \Gamma$ and using the boundary conditions in Eq. (1.2), a singular integrodifferential equation for the determination of the function

$$\mu(t) = T^+ - T^- \tag{1.3}$$

is obtained

$$|t'(s)|\frac{\mu(t)}{\delta(s)} = \mathrm{Re}\left[t'(s)\left(iF'(t) + \frac{1}{2\pi}\int_{\Gamma}\frac{\mu'(\tau)\,d\tau}{\tau - t}\right)\right]. \tag{1.4}$$

Here s is an arbitrary increasing parameter such that, when s varies over the interval $[s^-, s^+]$, the point t(s) covers the whole length of Γ. Suppose that the limiting temperature values T^+ and T^- at each end of the line Γ_n are equal; then, from Eq. (1.3)

$$\mu(t_n^-) = \mu(t_n^+) = 0. \tag{1.5}$$

Here t_n^- and t_n^+ denote the left-hand and right-hand ends of the line Γ_n, respectively. The condition in Eq. (1.5) must be used in solving Eq. (1.4).

If the real part of the given function F(z) and the system of inclusions Γ are biperiodic with basic periods 2ω and $2\omega'$, the temperature field perturbed by inclusions will also be biperiodic, i.e., the real part Re W(z) of the complex potential of the perturbed field is a biperiodic function [the imaginary part of F(z) or W(z) may differ by a constant value at congruent points]. To construct this periodic function it is necessary to sum the Cauchy-type integrals of the form in Eq. (1.1) taken over the whole length $t + 2n\omega + 2m\omega'$, where $t \in \Gamma$, while n and m are integers. The Weierstrass zeta function $\zeta(u)$ may be used [2, 3], and the complex potential may be written in the form

$$W(z) = F(z) + \Phi(z) + Cz,$$
$$\Phi(z) = \frac{1}{2\pi i}\int_{\Gamma}\mu(t)\vartheta\left(\frac{t-z}{2\omega}\right)dt, \quad \vartheta\left(\frac{u}{2\omega}\right) = \zeta(u) - \frac{u}{\omega}\zeta(\omega). \tag{1.6}$$

The unknown complex number C is determined by the biperiodicity condition for the temperature field

$$\mathrm{Re}\,[W(z+2\omega) - W(z)] = 0, \quad \mathrm{Re}\,[W(z+2\omega') - W(z)] = 0. \tag{1.7}$$

Since at congruent points the function $\vartheta(v)$ satisfies the relation

$$\vartheta(v + n + m\tau) = \vartheta(v) - \pi m i/\omega, \quad \tau = \omega'/\omega, \tag{1.8}$$

where n and m are integers, Eqs. (1.6) and (1.7) lead to a condition uniquely determining the constant C

$$\mathrm{Re}\,[\omega C] = 0, \quad \mathrm{Re}\left[4\omega\omega' C + \int_{\Gamma}\mu(t)\,dt\right] = 0. \tag{1.9}$$

It follows from these relations that if

$$\int_{\Gamma}\mu(t)\,dt = 0, \tag{1.10}$$

then C = 0 and $\Phi(z)$ in Eq. (1.6) is biperiodic; otherwise, the imaginary part of $\Phi(z)$ at congruent points would not be a constant.

To determine the real function $\mu(t)$, Eq. (1.6) yields, in view of the boundary conditions in Eq. (1.2), the equation

$$|t'(s)|\frac{\mu(t)}{\delta(s)} = \operatorname{Re}\left[t'(s)\left(iF'(t) + iC + \frac{1}{2\pi}\int_{\Gamma}\mu'(\tau)\,\vartheta\left(\frac{\tau - t}{2\omega}\right)d\tau\right)\right]. \quad (1.11)$$

Remark. Passing to the limit as $\omega' \to \infty$ in Eqs. (1.6) and (1.11) yields the expression for the complex potential and the equation for $\mu(t)$ in the periodic case, considered in [4].

2. Suppose that in the plane considered above there is an infinite system of infinite parallel rectilinear cuts (inclusions) of low conductivity and constant width (δ = const) at a constant spacing of 2d. The axis 0x in the z plane is directed along one of the inclusions. Suppose that a biperiodic function with periods $2a$ and 2di is defined in the z plane. Then the temperature field perturbed by inclusions will be biperiodic with periods $2a$ and 2di. As the basic parallelogram, the rectangle $D = \{-a \le x < a,\ -d \le y < d\}$ will be used. The line Γ coincides with the segment of the real axis $-a \le x \le a$.

Since the function $\vartheta(v)$ takes real values for real values of the argument, Eq. (1.11) may be written for the given case, using the boundary values of $\Phi(z)$ in Eq. (1.8), in the form

$$\Phi^+ - i\delta\Phi^{+\prime}/2 = \Phi^- + i\delta\Phi^{-\prime}/2 + \delta\operatorname{Re}[iF'(x) - C_0]. \quad (2.1)$$

The subscripts + and − denote the limiting values of $\Phi(z)$ and $\Phi'(z)$ above and below the integration line $-a \le x \le a$, respectively. The constant C_0 is determined, in accordance with Eq. (1.9), from the formula

$$C_0 = \frac{1}{4ad}\int_{-a}^{a}\mu(x)\,dx,\ C = iC_0. \quad (2.2)$$

It will be expedient to introduce the auxiliary function

$$\Psi^{\pm}(x) = \Phi^{\pm}(x) \mp i\delta\Phi^{\pm\prime}(x)/2. \quad (2.3)$$

The boundary condition in Eq. (2.1) then takes the simple form

$$\Psi^+ - \Psi^- = \delta\operatorname{Re}[iF'(x) - C_0]. \quad (2.4)$$

Since $\Phi(z)$ and $\Phi'(z)$ are written using integrals with the kernel $\vartheta(v)$ and satisfy the conditions

$$\overline{\Phi}(z) = -\Phi(z),\quad \overline{\Phi}'(z) = -\Phi'(z),\quad (\overline{\Phi}(z) = \overline{\Phi(\bar{z})}),$$

the functions $\Psi^{\pm}(x)$ are boundary values of the piecewise holomorphic function $\Psi(z)$, satisfying the condition $\overline{\Psi}(z) = -\Psi(z)$ and written using an integral with kernel $\vartheta(v)$. Therefore, on the basis of the Sokhotskii formula [2], the following expression may be written:

$$\Psi(z) = \frac{\delta}{2\pi i}\int_{-a}^{a}\operatorname{Re}[iF'(\xi) - C_0]\,\vartheta\left(\frac{\xi - z}{2a}\right)d\xi. \quad (2.5)$$

Using the boundary values of $\Psi(z)$ in Eq. (2.5) and integrating Eq. (2.3), the result obtained for the function $\mu(x) = \Psi^+ - \Phi^-$, taking into account the periodicity of the boundary values $\Phi^{\pm}(x)$, is

$$\mu(x) = \frac{2i}{\delta}\int^{x}\left\{\Psi^-(\xi)\exp\left[\frac{2i(\xi - x)}{\delta}\right] + \Psi^-(\xi)\exp\left[\frac{2i(x - \xi)}{\delta}\right]\right\}d\xi. \quad (2.6)$$

Substituting $\mu(x)$ from Eq. (2.6) into Eqs. (1.6) and (2.2), the function $\Phi(z)$ and the unknown constant C are found.

For example, suppose that in the considered plane sources of equal strength $q > 0$ and sinks of strength $-q$ are introduced, respectively, at the points: a) $z = (2n + 1)a + i(2md + b)$, $z = (2n + 1)a + i[(2m + 1)d + b]$ (a linear system of sources and sinks); b) $z = 2na + i(2md + b)$, $z = (2n + 1)a + i[(2m + 1)d + b]$ (a staggered arrangement of sources and sinks); n and m are integers; $0 < b \le d$. Then the complex potential of the unperturbed field may be written in the form

$$\text{a)}\ F(z) = \frac{q}{2\pi}\ln\frac{\vartheta_3(v)}{\vartheta_2(v)};\quad \text{b)}\ F(z) = \frac{q}{2\pi}\ln\frac{\vartheta_3(v)}{\vartheta_1(v)};\quad v = \frac{z - ib}{2a}, \quad (2.7)$$

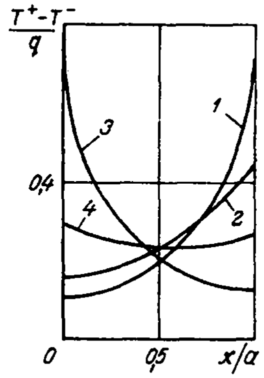

Fig. 1. Dimensionless temperature drop along inclusions for different values of b/a.

$\vartheta_k(v)$, k = 1, 2, 3, are the first, second, and third theta functions [2, 3]. Below, consideration will be limited to the basic rectangle $D = \{-a \le x < a, -d \le y < d\}$. The functions $\vartheta(v)$ and $F'(x)$ may then be written in a form more convenient for calculation [3]

$$\vartheta\left(\frac{\xi-z}{2a}\right)=\frac{\pi}{2a}\operatorname{ctg}\frac{\pi(\xi-z)}{2a}+\frac{2\pi}{a}\sum_{n=1}^{\infty}\frac{h^{2n}}{1-h^{2n}}\sin\frac{n\pi(\xi-z)}{a}, \tag{2.8}$$

$$\operatorname{Re}[iF'(x)]=\frac{q}{a}\left[\frac{1}{4}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{a}\right],$$

$$\text{a)}\quad a_n=(-1)^n\left[\frac{1}{2\gamma^n}+\frac{1}{1+h^n}\operatorname{sh}\frac{n\pi b}{a}\right],\quad h=\exp\left(-\frac{\pi d}{a}\right), \tag{2.9}$$

$$\text{b)}\quad a_n=\frac{1}{2\gamma^n}-\frac{h^{2n}-(-1)^n h^n}{1-h^{2n}}\operatorname{sh}\frac{n\pi b}{a},\quad \gamma=\exp\left(\frac{\pi b}{a}\right). \tag{2.10}$$

Using Eqs. (2.8)-(2.10) and making the necessary computations in accordance with Eqs. (2.5), (2.6), (2.2), and (1.6), the following results are obtained:

$$\mu(x)=\frac{q\delta}{a}\left[\sum_{n=1}^{\infty}a_n b_n\cos\frac{n\pi x}{a}+\frac{1}{4(1+\delta/2d)}\right], \tag{2.11}$$

$$\Phi^{\pm}(z)=\frac{q\delta}{2a}\left[\sum_{n=1}^{\infty}a_n b_n\left(\frac{2ih^{2n}}{1-h^{2n}}\sin\frac{n\pi z}{a}\pm\exp\left(\pm\frac{in\pi z}{a}\right)\right)\pm 1/4(1+\delta/2d)\right],\quad C_0=q/4a(1+2d/\delta), \tag{2.12}$$

$$b_n=[1-h^{2n}-(1+h^{2n})\,n\pi\delta/2a]/(1-h^{2n})[1-(n\pi\delta/2a)^2].$$

In Eq. (2.12) the upper (+) and lower (-) signs refers, respectively, to the values $0 < y \le d$ and $-d \le y < 0$; the coefficients a_n are determined from Eq. (2.10).

In Fig. 1, curves of the temperature drop (as a function of q) along the inclusions are shown for $0 \le x/a \le 1$ with $d/a = 1$, $\delta/2a = 1$, and: a) $b/a = 0.1$ (1), 0.5 (2); b) $b/a = 0.1$ (3), 0.5 (4).

NOTATION

T^+ and T^-	are the values of the temperature T at the left-hand and right-hand edges of the inclusion;
Ψ	is the current function;
F(z)	is the complex potential of temperature field unperturbed by inclusions;
W(z)	is the complex potential of temperature field perturbed by inclusions;

k_0 and k	are the thermal conductivity of the inclusions and the body;
Γ_n	is the smooth line in the complex z plane;
Γ	is the piecewise continuous line;
$2h_0$	is the width of the inclusion;
2ω and $2\omega'$	are the periods of complex potential W(z);
q	is the source strength.

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